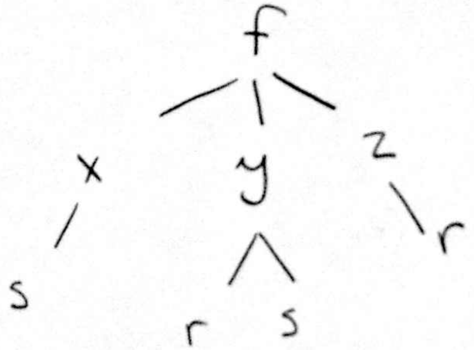
Section 15.6 WKST Solutions

1. Use the chain rule to calculate the partial derivatives $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial r}$ for $f(x,y,z) = xy + z^2$, with

$x = s^2$, $y = 2rs$, $z = r^2$.



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= y \cdot 2s + x \cdot 2r$$

$$= 2rs \cdot 2s + s^2 \cdot 2r$$

$$= 4rs^2 + 2rs^2$$

$$= \boxed{6rs^2}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= x \cdot 2s + 2z \cdot 2r$$

$$= \boxed{2s^3 + 4r^3}$$

2. Let $u = u(x,y)$ and let (r, θ) be polar coordinates

Verify the relation $\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2} u_\theta^2$.

We have $x = r\cos\theta$ and $y = r\sin\theta$, then

$u(x,y) = u(r\cos\theta, r\sin\theta)$.

Then, by the chain rule,

$$u_r = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= u_x \cdot \cos\theta + u_y \cdot \sin\theta.$$

$$u_\theta = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= -u_x r\sin\theta + u_y r\cos\theta.$$

So, $u_r^2 + \frac{1}{r^2} u_\theta^2 = (u_x\cos\theta + u_y\sin\theta)^2 + \frac{1}{r^2}(-u_x r\sin\theta + u_y r\cos\theta)^2$

$$= u_x^2\cos^2\theta + 2u_xu_y\cos\theta\sin\theta + u_y^2\sin^2\theta + \frac{1}{r^2}(u_x r^2\sin^2\theta - 2u_xu_y r^2\sin\theta\cos\theta + r^2\cos^2\theta)$$

$$= u_x^2(\cos^2\theta + \sin^2\theta) + u_y^2(\sin^2\theta + \cos^2\theta)$$

$$= u_x^2 + u_y^2$$

$$= \|\nabla u\|^2. \checkmark$$

Brief Summary of 15.7

- Complete the following statement of the 2nd derivative test for $f(x,y)$. Let $P=(a,b)$ be a critical point of $f(x,y)$. Assume that f_{xx}, f_{yy} and f_{xy} are continuous near P. Then:

(i) If $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}^2(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.

(ii) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.

(iii) If $D < 0$, then f has a saddle point at (a,b).

(iv) If $D = 0$, the test is inconclusive.

Section 15.7 Additional Exercises

1. Find the critical points of the function $f(x,y) = x^2 + y^2 - xy + x$. Use 2^{nd} derivative test to determine if they are local minima, local maxima, or saddle points.

<u>Finding Critical Points</u>

$$\begin{aligned} f_x(x,y) &= 2x - y + 1 = 0 \\ f_y(x,y) &= 2y - x = 0 \end{aligned} \Rightarrow 2y = x \Rightarrow 2(2y) - y + 1 = 0$$

$$\Rightarrow y = -1/3 \Rightarrow x = -2/3$$

So, the only critical point is $(-2/3, -1/3)$.

Second derivative test

$f_{xx}(x,y) = 2$

$f_{yy}(x,y) = 2 \implies D = 4 - (-1)^2 = 4 - 1 = 3 > 0$

$f_{xy}(x,y) = -1$ and $f_{xx}(-2/3, -1/3) = 2 > 0$

$\implies \boxed{(-2/3, -1/3) \text{ is a local min.}}$

2. Repeat the above for $f(x,y) = xy\, e^{-x^2-y^2}$.

Finding critical points

$$f_x(x,y) = y e^{-x^2-y^2} + xy(-2x) e^{-x^2-y^2}$$

$$= y\, e^{-x^2-y^2}(1-2x^2)$$

$$f_y(x,y) = x e^{-x^2-y^2}(1-2y^2).$$

Setting these equal to zero gives:

$$\begin{cases} y e^{-x^2-y^2}(1-2x^2) = 0 & (1) \\ x e^{-x^2-y^2}(1-2y^2) = 0 & (2) \end{cases}$$

$(1) \Rightarrow y=0$ or $1-2x^2=0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$(2) \Rightarrow x=0$ or $1-2y^2=0 \Rightarrow y=\pm\frac{1}{\sqrt{2}}$

So, the critical points are:

$$\boxed{(0,0),\ (1/\sqrt{2}, 1/\sqrt{2}),\ (-1/\sqrt{2}, 1/\sqrt{2}),\ (1/\sqrt{2}, -1/\sqrt{2}),\ (-1/\sqrt{2}, -1/\sqrt{2})}$$

$$f_{xx}(x,y) = -2xye^{-x^2-y^2}(1-2x^2) + e^{-x^2-y^2}(-4xy)$$
$$= -2xye^{-x^2-y^2}(1-2x^2+2) = -2xye^{-x^2-y^2}(3-2x^2)$$

$$f_{yy}(x,y) = -2xye^{-x^2-y^2}(3-2y^2)$$

$$f_{xy}(x,y) = (1-2x^2)\frac{\partial}{\partial y}(ye^{-x^2-y^2}) = (1-2x^2)\left[e^{-x^2-y^2} - 2y^2e^{-x^2-y^2}\right]$$
$$= (1-2x^2)(1-2y^2)e^{-x^2-y^2}.$$

So,

$$D(x,y) = 4x^2y^2(e^{-x^2-y^2})^2(3-2x^2)(3-2y^2) - (e^{-x^2-y^2})^2(1-2x^2)(1-2y^2)$$

$\hookrightarrow$

$$=\underbrace{\left(e^{-x^2-y^2}\right)^2}_{(*)}\underbrace{\left[4x^2y^2(3-2x^2)(3-2y^2)-(1-2x^2)(1-2y^2)\right]}_{(**)}.$$

Since $(*)>0$, the sign is determined by $(**)$.

Critical Point (a,b)	Sign of $D(a,b)$	Sign of $f_{xx}(a,b)$	type of critical point
$(0,0)$	negative	$\times$	saddle point
$(1/\sqrt{2}, 1/\sqrt{2})$	positive	negative	local max
$(-1/\sqrt{2}, -1/\sqrt{2})$	positive	negative	local max
$(1/\sqrt{2}, -1/\sqrt{2})$	positive	positive	local min
$(-1/\sqrt{2}, 1/\sqrt{2})$	positive	positive	local min

3. Determine the <u>global</u> extreme values of the function $f(x,y) = x^2 + 2y^2$ on the domain $0 \le x \le 1$, $0 \le y \le 1$.

First, we find the critical points in the interior

$f_x(x,y) = 2x$
$f_y(x,y) = 2y$ $\Longrightarrow$ $(0,0)$ is a critical point.

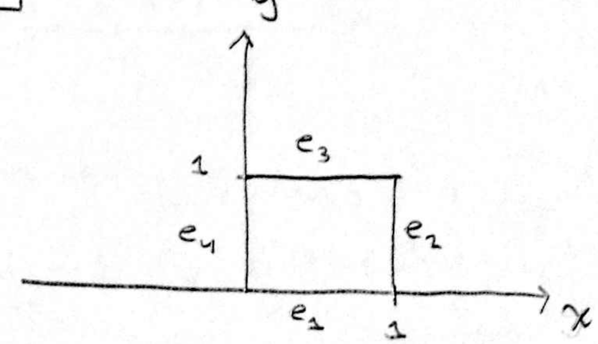
Second derivative test

$f_{xx} = 2$
$f_{yy} = 4$ $\Longrightarrow$ $D = 8 > 0$ and $f_{xx} > 0 \Longrightarrow$
$f_{xy} = 0$

$\boxed{f(0,0) \text{ is a local min}}$

testing the boundary

edge	f restricted to edge	max on edge	min on edge
e_1	$f(x,0) = x^2$	1	0
e_2	$f(1,y) = 1 + 2y^2$	3	1
e_3	$f(x,1) = x^2 + 2$	3	2
e_4	$f(0,y) = 2y^2$	2	0



So, min occurs @ $(0,0)$ and is 0, max occurs @ $(1,1)$ and is 3.

Summary 15.8

- Complete the statement of the Lagrange Multiplier theorem. Assume that $f(x,y)$ and $g(x,y)$ are differentiable functions. If $f(x,y)$ has a local minimum or a local maximum on the constraint curve $g(x,y)=0$ at $P=(a,b)$ and $\nabla g_P \neq 0$, then there is a scalar λ such that $\nabla f_P = \lambda \nabla g_P$.

- If the problem is to minimize/maximize $f(x,y,z)$ subject to the constraints $g(x,y,z)=0$ and $h(x,y,z)=0$, then the Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$.

Section 15.8 Additional Exercises

1. Find the minimum and maximum values of the function $f(x,y)=x^2+y^2$ subject to the constraint $2x+3y=6$.

Assume a critical point under the constaint exists. Call it (a,b). Then letting $g(x,y)=2x+3y-6$ we have:

$$\nabla f_{(a,b)} = \lambda \nabla g_{(a,b)} \quad \text{for some } \lambda.$$

We have $\nabla f_{(a,b)} = \langle 2a, 2b \rangle$ 6

$\nabla g_{(a,b)} = \langle 2, 3 \rangle. \implies$

$$\begin{cases} 2a = \lambda 2 \\ 2b = \lambda 3 \end{cases} \implies \begin{cases} a = \lambda \\ 2/_3 b = \lambda \end{cases} \implies a = \frac{2}{3} b$$

Plugging this into the constraint gives:

$$2\left(\frac{2}{3}b\right) + 3b = 6 \implies \frac{4b}{3} + \frac{9b}{3} = 6$$

$$\implies \frac{13b}{3} = 6$$

$$\implies b = \frac{18}{13}$$

$$a = \frac{2}{3} \cdot \frac{18}{13} = \frac{12}{13}.$$

Hence the only possible critical point is $(a,b) = \left(\frac{12}{13}, \frac{18}{13}\right)$.

Finally, we need to prove existence of a critical point and determine if it is a max/min.

the constraint implies $y = -2/_3 x + 2$ as $|x| \to \infty$ so does $|y|$ So, $f(x,y) = x^2 + y^2$ is increasing with out bound, on this constraint

$\implies$ $\boxed{f(12/_{13}, 18/_{13}) = \frac{468}{169} \approx 2.77 \text{ is a min. of } f \text{ on this constraint.}}$

2. Find the point (a,b) on the graph of $y=e^x$ where the value ab is as small as possible.

Let $f(a,b) = ab$ with constraint $b = e^a \Rightarrow 0 = e^a - b$.

Let $g(a,b) = e^a - b$.

If a min. exists, it must satisfy,

$$\nabla f_{(a_0,b_0)} = \lambda \nabla g_{(a_0,b_0)}.$$

$$\nabla f_{(a_0,b_0)} = \langle b_0, a_0 \rangle, \quad \nabla g_{(a_0,b_0)} = \langle e^{a_0}, -1 \rangle \Rightarrow$$

$$\begin{cases} b_0 = \lambda e^{a_0} \\ a_0 = -\lambda \end{cases} \Rightarrow \frac{b_0}{e^{a_0}} = -a_0$$

Plugging this into the constraint gives:

$$e^{a_0} - (-a_0 e^{a_0}) = 0 \Rightarrow e^{a_0}(1 + a_0) = 0$$

$$\Rightarrow a_0 = -1$$

$$\Rightarrow b_0 = e^{-1}.$$

So, if a min. occurs, it is @ $(-1, e^{-1})$ and $f(-1, e^{-1}) = -e^{-1}$

Since the constraint is not bounded, we need to justify the existence of a minimum value.

8

The values $f(x,y) = xy$ on the constraint $y = e^x$ are $f(x, e^x) = h(x) = xe^x$. Since $h(x) > 0$ for $x > 0$, the minimum value if exists, occurs at a point $x < 0$

Since $\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} -e^x = 0$

We have For any $\epsilon > 0$ $\exists$ $-R$ s.t.

$$x < -R \Rightarrow |h(x)| < \epsilon$$

Take $\epsilon = 0.0001 \Rightarrow$ $x < -R$ $|h(x)| < 0.0001$.

So, on the closed and bounded region $-R \leq x \leq 0$ f has a minimum. So, $-e^{-1} \approx -0.37$ is a global minimum.

3. Find the maximum value of $f(x,y) = x^a y^b$ for $x \geq 0$ $y \geq 0$ on the line $x + y = 1$, where $a, b, c \geq 0$ are constant

Let $g(x,y) = x + y - 1$ and assume (x_0, y_0) is the desired critical point. Then, we must have

$$\nabla f(x_0, y_0) = \langle a x_0^{a-1} y_0^b , b_0 y_0^{b-1} x_0^a \rangle$$

$$\nabla g(x_0, y_0) = \langle 1, 1 \rangle$$

9

So, we have
$$\begin{cases} a x_0^{a-1} y_0^{b} = \lambda \\ \\ b_0 y_0^{b-1} x_0^{a} = \lambda \end{cases}$$

Note if $\lambda = 0$ we get $(x_0, y_0) = (0,0)$ is a critical point which corresponds to the min in this case

$$\Longrightarrow \quad a x_0^{a-1} y_0^{b} = b y_0^{b-1} x_0^{a}$$

$$\Longrightarrow \quad a y_0 = b x_0 \Rightarrow y_0 = \frac{b}{a} x_0 .$$

Plugging into the constraint, this gives us:

$$x_0 + \frac{b x_0}{a} = 1 \Longrightarrow \frac{x_0(a+b)}{a} = 1 \Longrightarrow$$

$$x_0 = \frac{a}{a+b}$$

$$y_0 = \frac{ba}{a(a+b)} = \frac{b}{a+b}$$

So, if a max value occurs on the constraint, then it occurs at $(x_0, y_0) = \left(\frac{a}{a+b}, \frac{b}{a+b}\right)$ and it is equal to $f\left(\frac{a}{a+b}, \frac{b}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b = \frac{a^a b^b}{(a+b)^{a+b}}$

Since the constraint $x + y = 1$, $x \geq 0$, $y \geq 0$ is clearly closed and bounded, a min. and max of f must occur on the constraint.